Reading seminar on moduli space of boundary polarised CY pairs

Plan:

1. Explain the motivation including

   - conjectural existence of good moduli space for certain CY pairs to interpolate K-moduli space and KSBA moduli space.
   - Potential application to B-semiampleness conjecture in birational geometry following the suggestion of Shokurov.

   and state the main theorems in [2].

2.
   - Define boundary polarised CY pairs and review the slc singularities.
   - Provide examples, including plane curve pairs, del pezzo surface pairs etc.
   - Define the moduli stack
   $$\mathfrak{M}^{CY}_{\chi,\overrightarrow{r},N} : \; \mathcal{S}ch \; \longrightarrow \; \mathcal{S}et$$
   for boundary polarised CY pairs with boundadry coefficients $\overrightarrow{r}$, Hilbert polynomial χ and index N. That is, define the family of pairs over a base scheme. One may follow [3] or [2, Section 2,3.1,3.2]. A key notion for family of pairs is the relative Mumford divisor introduced by Kollar [3].
   - Explain $\mathfrak{M}^{CY}_{\chi,\overrightarrow{r},N}$ is an algebraic stack of locally finite type with affine diagonal. Show $\mathfrak{M}^{CY}_{\chi,\overrightarrow{r},N}$ is not finite type in general by plane curve example. This is main difficulty for constructing a good moduli space for CY pairs.

3. Introduce moduli theory tool: good moduli space theory and theta-stratification theory on Artin stack due to Jarod Alper, Daniel Halpern-Leistne etc. This includes

   - Explain what is good moduli space and its basic properties.
   - Explain Θ-reductivity and S-completeness.
   - State the existence criterion in [1] for good moduli space.

4. Translate the Θ-reductivity and S-completeness for the moduli stack $\mathfrak{M}$ to the family over $\overline{ST_R}$ and Θ_R and explain why the moduli stack $\mathfrak{M}^{CY}_{\chi,\overrightarrow{r},N}$ satisfies Θ-reductivity and S-completeness. And prove the valuative criterion of properness.

5. Explain some general attempts in [2] to construct good moduli space for boundary polarised CY pairs using coregularity.

6. Discuss the case $\mathcal{P}^{CY}_d \subset \mathfrak{M}^{CY}_{\chi_d,\overrightarrow{r}_d,N_d}$ for plane curves of degree d.

   (a) Show the substack $\mathcal{P}^{CY}_{d,m} \subset \mathcal{P}^{CY}_d$ is finity type and admits good moduli space for any large m.

(b) Show $\mathcal{P}_{d,m}^{CY}$ stablizes for $m \gg 0$ and the existence of asymtotical good moduli space P_d^{CY}, serving as semi-normalization of good moduli space $P_{d,m}^{CY}$.

7. Explain the application to B-semiampleness conjecture in dimension 2.

    (a) Explain general strategy to the B-semiampleness conjecture.

    (b) Prove ampleness of Hodge line bundles over substack of $\mathfrak{M}_{\chi, \overrightarrow{r}, N}^{CY}$, which admits proper good moduli space under certain condition.

Place: Quanzhai;   **Time:** Tuesday

References

[1] Jarod Alper, Daniel Halpern-Leistner, and Jochen Heinloth. Existence of moduli spaces for algebraic stacks. *Inventiones mathematicae*, pages 1–90, 2023.

[2] Kenneth Ascher, Dori Bejleri, Harold Blum, Kristin DeVleming, Giovanni Inchiostro, Yuchen Liu, and Xiaowei Wang. Moduli of boundary polarized calabi-yau pairs. *arXiv preprint arXiv:2307.06522*, 2023.

[3] János Kollár. *Families of varieties of general type*, volume 231 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2023. With the collaboration of Klaus Altmann and Sándor J. Kovács.